Polyhedron arithmetic

Having built a polyhedron from a net or illustration, students will feel pleased with what they have done. But they can now capitalise on their success by finding the properties of their shape. Most important are the planes of mirror symmetry and axes of rotation symmetry. Then there is the matter of how many faces, edges and vertices they have. It is this second question which concerns us here.

On a complicated shape it is even difficult to count the faces. It is usually necessary to number them on a net. But there is simple and interesting combinatorial mathematics to be learned in finding the face (f), edge (e) and vertex (v) numbers without the help of the solid or its net. The algebra is well within range for international Grade 8/British Y9 pupils.

I'm taking all my examples from the Platonic and Archimedean solids. I'm doing that because, in those, *every vertex is the same*: around a vertex the same shapes meet in the same order. And, amongst them, I'm taking just those with pentagonal faces. I'm doing that because of the suprising property that *any* polyhedron which does not have a hole through it and contains pentagons, must have exactly 12 !

From that starting point we can find out everything we need to know.

In each case, though we may need to find the number of edges and faces along the way, my aim will be to find the number of vertices. I'm doing that because we can do so in no fewer than three distinct ways. Here they are:

1. The 'times-and-share' principle.

This is best explained through our examples.

But I'm first going to define what I shall call the 'loose shape number' (*l.s.n*). Take the regular dodecahedron, which has 12 pentagonal faces. If I have 12 loose pentagons on the table, they have a total of 12 x 5 = 60 vertices and the same number of edges. That is because an individual polygon has the same number of vertices as edges. I'm then going to say the *l.s.n.* for the dodecahedron is 60. You'll see in the examples how we make use of this fact.

2. Euler's formula: $v + f = e + 2$.

Making v the subject, we have: $v = e + 2 - f$.
As you see, we need to find e and f before we have v.

3. Descartes' rule.

Working in degrees:
We find the amount by which the sum of the angles at a vertex, s, falls short of 360, the *angle defect*, $d = 360 - s$.
We then divide d into 720: $v = \frac{720}{d}$.

> The idea here is that a solid with very flat vertices, like the football, needs a lot of them to close completely, whereas a solid with very pointed vertices, like the tetrahedron, needs few.

We hope of course that each method will yield the same v. We shall make a table with one column for each method. We shall set columns **1** and **2** alongside because we shall need to feed information from column **1** to column **2**.

We shall use a suffix when we need to distinguish different shapes. So, for example, we shall write ‘ f_5’ for the number of pentagons. For column **3** the students need to know the interior angles of the regular 3-, 4-, 5- and 6-gons.

The regular dodecahedron, 5^3

1.	**2.**	**3.**
l.s.n. = 12 x 5 = 60. But 3 faces share each vertex. Therefore $v = \frac{60}{3} = \mathbf{20}$.	*l.s.n.* = 60. But 2 faces share each edge. Therefore $e = \frac{60}{2} = 30$. $f = 12$. $v = e + 2 - f$ $= 30 + 2 - 12 = \mathbf{20}$.	$s = 108 \text{ x } 3 = 324$. $d = 360 - 324 = 36$. $v = \frac{720}{36} = \mathbf{20}$.

The icosidodecahedron, 3.5.3.5

1.	**2.**	**3.**
There are 12 pentagons: $f_5 = 12.$ There are 5 triangles to every pentagon, but 3 pentagons to every triangle. So the number of triangles, $f_3 = \frac{12 \times 5}{3} = 20.$ *l.s.n.* = $(f_5 \times 5) + (f_3 \times 3) =$ (12 x 5) + (20 x 3) = 120. But 4 faces share a vertex. Therefore $v = \frac{120}{4} = \mathbf{30.}$	$f = f_5 + f_3$ $= 12 + 20 = 32.$ *l.s.n.* = 120. But 2 faces share an edge. Therefore $e = \frac{120}{2} = 60.$ $v = e + 2 - f$ $= 60 + 2 - 32$ $= \mathbf{30}.$	$s = (2 \times 108) + (2 \times 60)$ $= 336.$ $d = 360 - 336 = 24.$ $v = \frac{720}{24} = \mathbf{30.}$

The rhombicosidodecahedron, 3.4.5.4

1.	**2.**	**3.**
f_5 = 12. There are 5 squares to every pentagon, but 2 pentagons to every square. So the number of squares, $f_4 = \frac{12 \times 5}{2}$ = 30. There are 2 triangles to every square, but 3 squares to every triangle. So the number of triangles, $f_3 = \dfrac{30 \times 2}{3} = 20.$ So *l.s.n.* = $(f_5{\times}5) + (f_4{\times}4) + (f_3{\times}3)$ = (12x5) + (30x4) + (20x3) = 240. But 4 faces a vertex. Therefore $v = \frac{240}{4} = \mathbf{60}$.	$f = f_5 + f_4 + f_3$ = 12 + 30 + 20 = 62. *l.s.n.* = 240. But 2 faces share an edge. Therefore $e = \frac{240}{2} = 120.$ $v = e + 2 - f$ = 120 + 2 – 62 = **60**.	*s* = 108 + (2 x 90) + 60 = 348. *d* = 360 – 348 = 12. $v = \frac{720}{12} = \mathbf{60}$.

The snub dodecahedron, $3^4.5$

1.	**2.**	**3.**
$f_5 = 12.$ 5 triangles share an edge with every pentagon, so we have 12 x 5 = 60 of those: $f_{3A} = 12 \times 5 = 60.$ But we also have another set of triangles. 5 of these share a vertex with every pentagon, but 3 pentgons share 1 of these triangles, so we have $f_{3B} = \frac{12 \times 5}{3} = 20$ of those. So *l.s.n.* = (12x5) + ([60+20]x3) = 300. But 5 faces share a vertex so $v = \frac{300}{5}$ = **60**.	$f = f_5 + f_{3A} + f_{3B}$ $= 12 + 60 + 20$ $= 92.$ *l.s.n.* = 300. But 2 faces share an edge so $e = \frac{300}{2}$ = 150. $v = e + 2 - f$ = 150 + 2 – 92 = **60**.	*s* = 108 + (4 x 60) = 348. *d* = 360 – 348 = 12. $v = \frac{720}{12} = \mathbf{60}.$

The truncated icosahedron, 5.6^2

1.	2.	3.
$f_5 = 12.$ There are 5 hexagons to every pentagon, but 3 pentagons share a hexagon. So $f_6 = \frac{12 \times 5}{3} = 20.$ *l.s.n.* = $(f_5{\times}5) + (f_6{\times}6)$ = (12 x 5) + (20 x 6) = 180. But 3 faces share a vertex so $v = \frac{180}{3} = \mathbf{60}.$	$f = f_5 + f_6$ = 12 + 20 = 32. *l.s.n.* = 180. But 2 faces share an edge. So $e = \frac{180}{2} = 90.$ $v = e + 2 - f$ = 90 + 2 – 32 = **60**.	s = 108 + (2 x 120) = 348. d = 360 – 348 =12. $v = \frac{720}{12} = \mathbf{60}.$

One fact may strike you. We always divide the *l.s.n.* by 2 to get e, and by the number of faces meeting in a vertex – which is the same as the number of edges meeting there – to get v. There must therefore be a simple relation between e and v. If the number of faces/edges meeting in a vertex is n:

$$2e = nv.$$

For the cube, for example, $2e = 3v$. (Check to see if I am right.)
For the regular octahedron, $2e = 4v$, so $e = 2v$.
For the regular icosahedron, $2e = 5v$.

You may like to substitute in Euler's formula to see what relations it yields.
(And again, check that your formulas work in simple cases.)

Paul Stephenson, 11.2.17