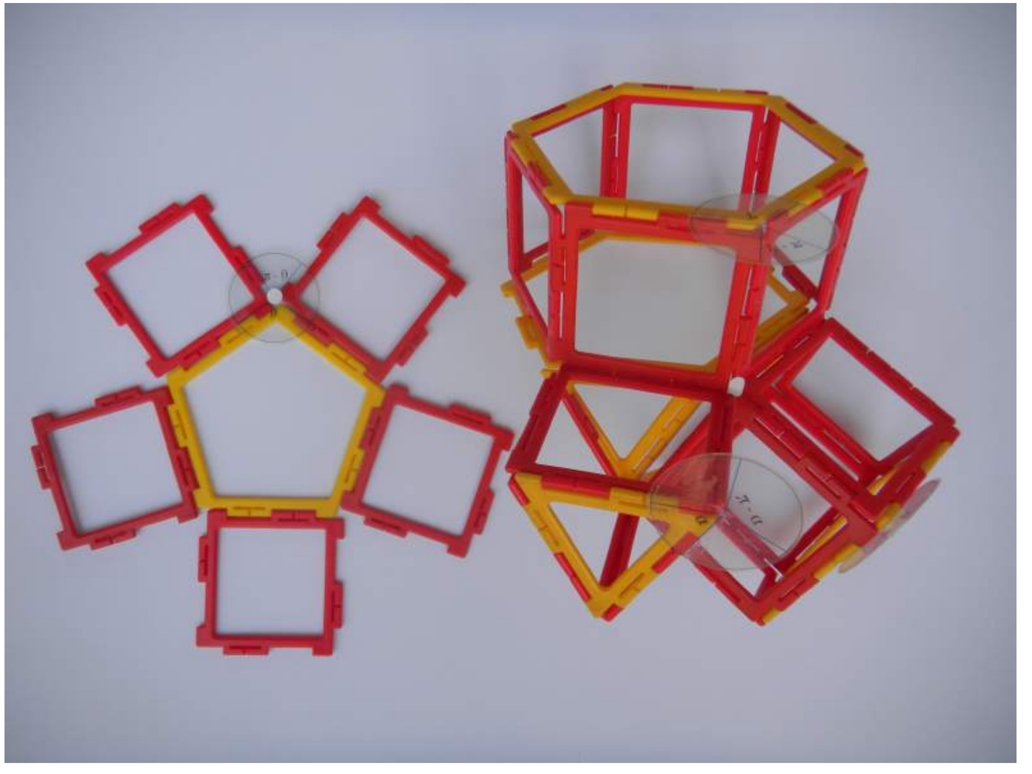
George Pólya’s reconstruction of an analogy Descartes may have drawn



On the left:

On the right:

Build *rectangles* on the *sides* of the poly*gon*.
The *circle* sectors between them fit together to complete a *circle*.

A *plane* angle can be stated in *radians* by quoting the *length* of *arc* of the unit *circle* subtending it.
A whole *plane* angle is 2π radians.

Build *prisms* on the *faces* of the poly*hedron*.
The *sphere* sectors between them fit together to complete a *sphere*.

A *spherical* angle can be stated in *ste*radians by quoting the *area* of *surface* of the unit *sphere* subtending it.
A whole *solid* angle is 4π *ste*radians.

The vertex model shows that the spherical triangle representing the angle defect, the amount by which a vertex falls short of a ‘flat’ angle, has angles of $A = \pi - \alpha$, $B = \pi - \beta$, $C = \pi - \gamma$. (*A*,*B*,*C* are the dihedral angles between the faces, α, β, γ the face angles.)

The total angle defect is a whole angle, 4π *ste*radians.

We shall use the sphere model to determine the area of our spherical triangle, and hence the angle defect it represents.

The white dots mark the vertices of the spherical triangle whose area Δ we require. Completing the great circle for each side, we find we've produced a congruent triangle in the antipodal position, shown by yellow dots.

Corresponding vertices mark the ends of lunes. We have a blue lune and a congruent, vertically opposite, blank lune. Likewise for red and green. Take the angle A in our triangle. The area of the lune to which it belongs is that fraction of 2π x the surface area of the sphere, $\frac{A}{2\pi} \times 4\pi = 2A$. Adding the corresponding blank lune, the area is double that, $4A$. The lunes of A, B and C and their doubles together cover the whole sphere, and the triangle and its double each 3 times. Thus the total exceeds 4π by the area of 4 triangles:

$4A + 4B + 4C = 4\pi + 4\Delta$, whence $\Delta = A + B + C - \pi$.

Since $A = \pi - \alpha$, $B = \pi - \beta$, $C = \pi - \gamma$, the area of our spherical triangle is $(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) - \pi = 2\pi - (\alpha + \beta + \gamma)$. This gives us a representation of the angle defect in terms of plane angles, (the angles of the faces meeting in the vertex), and a total angle defect of 4π radians.

Returning to our analogy,

On the left:

The vertex model shows that the angle defect, the amount by which a vertex falls short of a *straight* angle, is $\pi - \theta$ radians.

The total angle defect is 2π radians.

On the right:

We now know that the angle defect, the amount by which a vertex falls short of a *flat* angle, is $2\pi - (\alpha + \beta + \gamma)$ radians.

The total angle defect is 4π radians.