Map-colouring with Polydron

The 4 Colour Map Theorem says that you never need more than 4 colours to colour a map so that regions with the same colour don't touch. You have to count the region round the edge because the theorem is really about a map drawn on a sphere. The theorem is shape-blind. It doesn't matter what shape a region is. What matters are the regions it shares a border with.

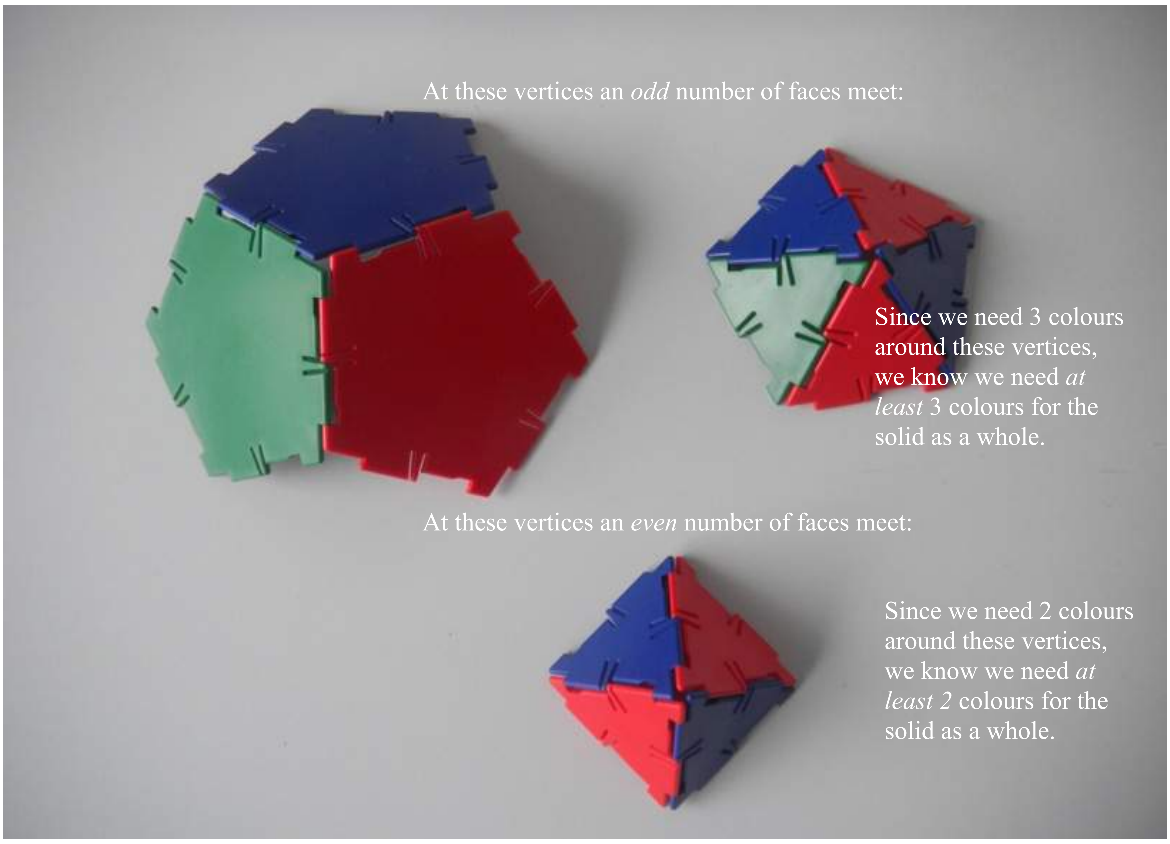
If the shapes are polygons so that the sphere becomes a polyhedron, colouring the map can still be a challenge and the Polydron pieces in their 4 standard colours are made for the job. The Framework shapes are best because you can see through to the other side.

If you've built the *Platonic* solids, where the polygons are regular and all the same, and the *Archimedean* solids, where the polygons are regular and not all the same but arranged the same way round each vertex, you're ready for these tasks.

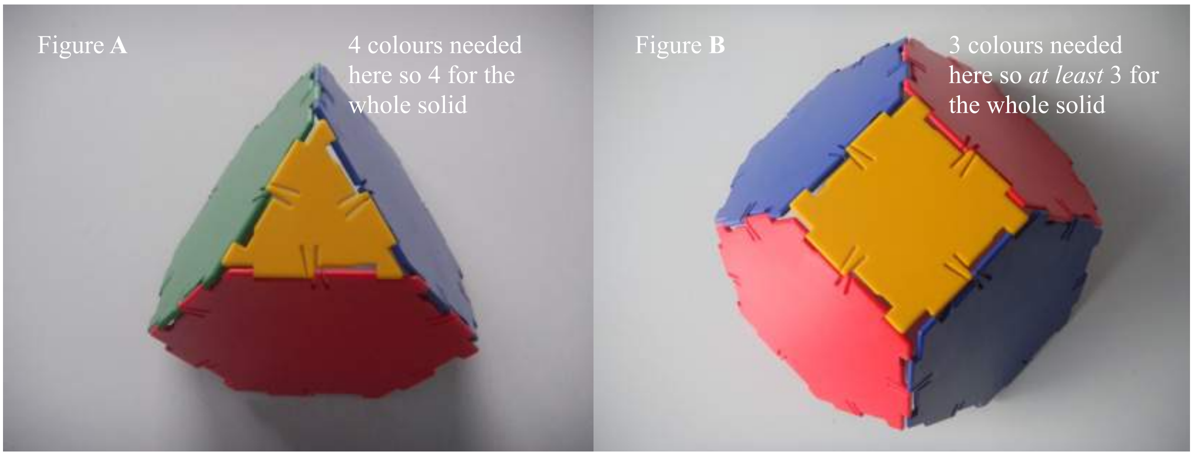
It will help always to ask ourselves these questions:

1. What *must* the *least* number of colours be?
2. Can I use symmetry to help me?
3. Can I use what I learned from one task to help me with another?

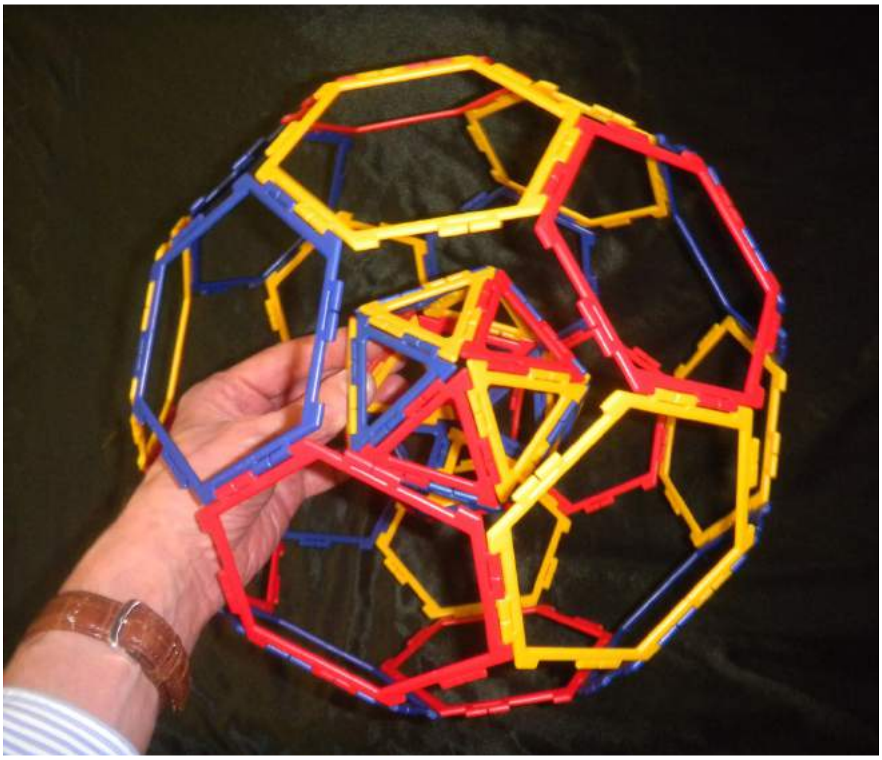
I imagine us building a solid in just one colour and rolling it around in our hands. We shall need to make *local* observations, involving a single vertex, and *global* observations, where we look at the solid as a whole. The local observation tells us only the *least* number needed. We may get away with using just that number or we may need one more.



If we make a slice through a vertex (we *truncate* the solid), the new face must have a different colour from all those round it, which increases the number of colours needed at that vertex by 1:



Case **A** includes all the following solids, which therefore require 4 colours: the **truncated tetrahedron, $\mathbf{3.6^2}$**, the **truncated cube, $\mathbf{3.8^2}$**, the **truncated dodecahedron, $\mathbf{3.10^2}$**, and the **truncated icosahedron, $\mathbf{5.6^2}$**. This picture shows the last one (the football) with the original solid inside, showing how, what were triangles and are now hexagons, can retain their original colours:

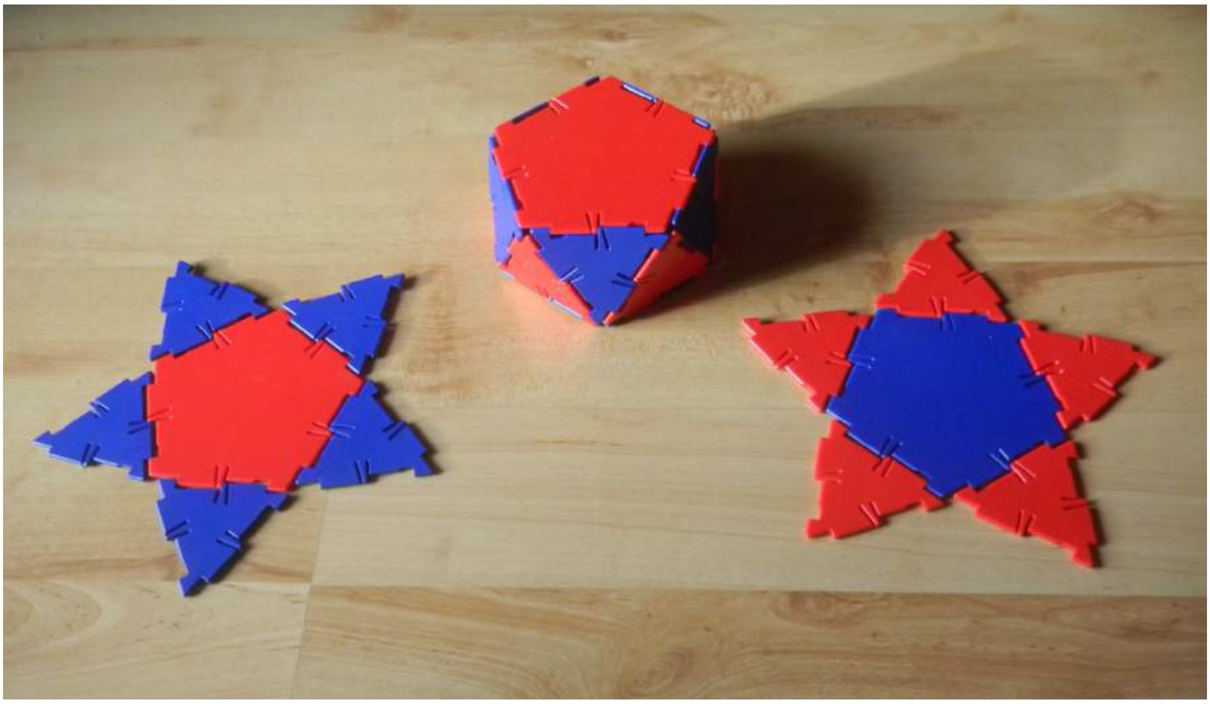


These diagrams depict not just truncated solids but any where a face is surrounded by faces which themselves touch.

For example, in figure **A** we could be looking at the **regular tetrahedron, 3^3**, whose net is shown below left, or the **regular dodecahedron, 5^3**, part of which is shown below right …

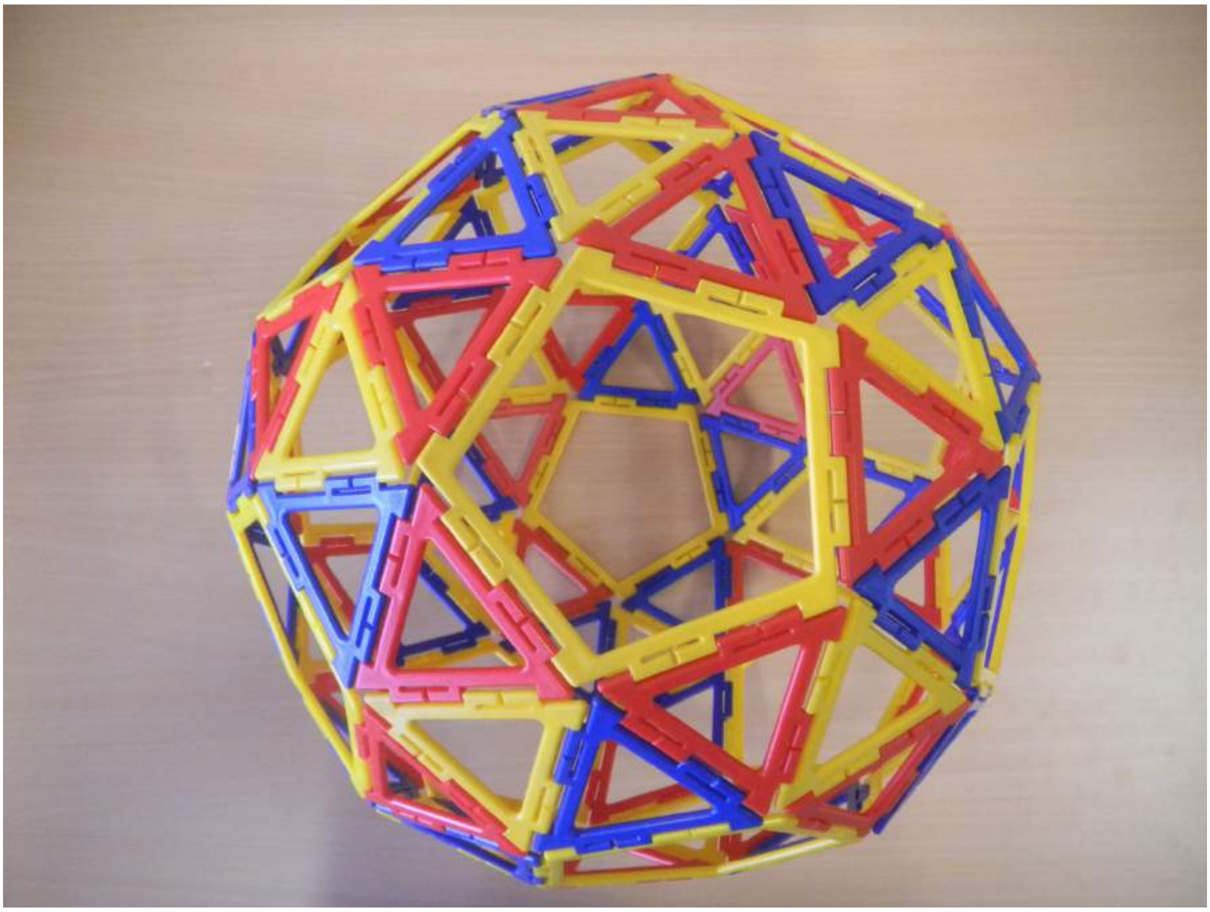
or at a **prism** whose top and bottom faces have an odd number of sides.

In figure **B** we could be looking at a prism whose top and bottom faces have an *even* number of sides. These include the **cube,** $\mathbf{4^3}$. In truncated solids the new faces are isolated from each other so, if we need 2 colours for the original solid, we shall need no more than 3 for the truncated form: we just give them all the 3rd colour. An example is the **truncated octahedron,** $\mathbf{4.6^2}$. To see why the **octahedron** itself, $\mathbf{3^4}$, only needs 2 colours, think of it as a triangular **antiprism**. In an antiprism we fit together two congruent pieces and just flip the colours used in one, for the colours used in the other. As you see, the number of sides the top and bottom faces have doesn't matter.



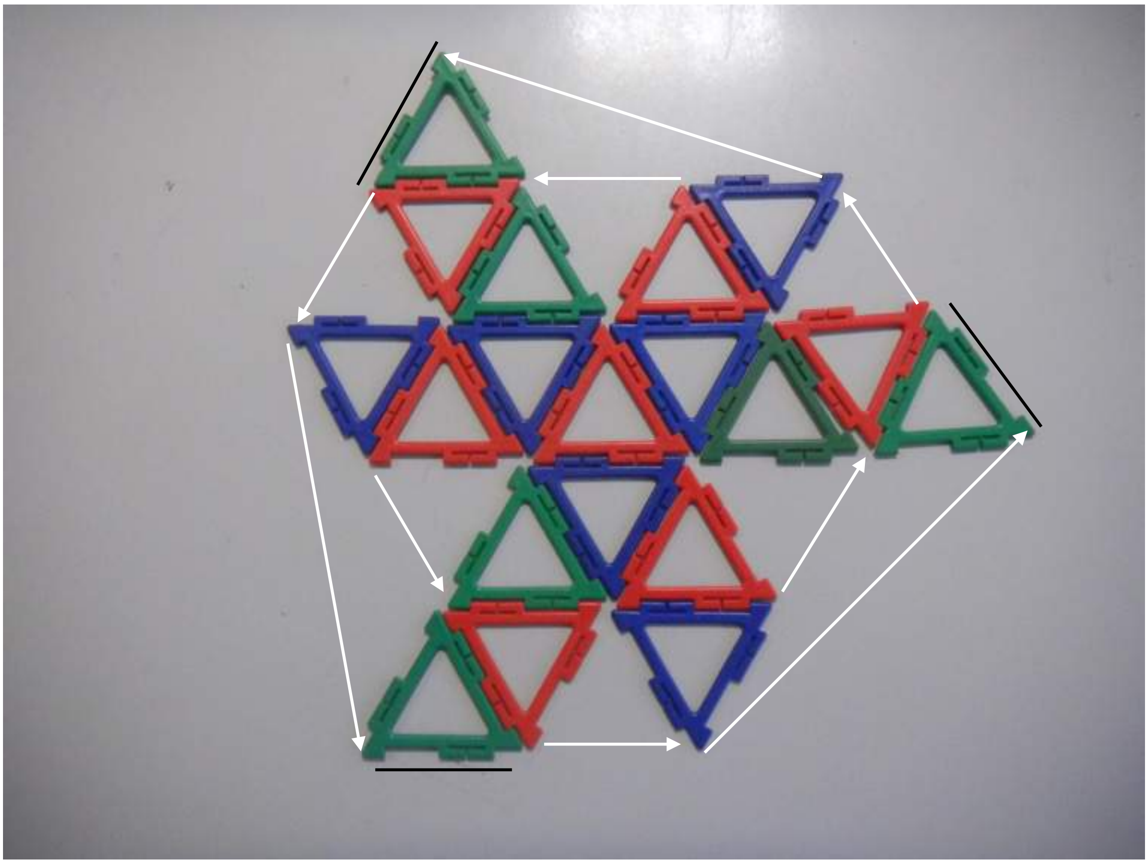
In the **cuboctahedron, 3.4.3.4** no triangle touches another triangle and no square touches another square. Each may therefore receive its own colour, giving a total of 2. The same applies to the two constituent face types of the **icosidodecahedron, 3.5.3.5**, and the three of the **rhombicosidodecahedron, 3.4.5.4**, the **truncated cuboctahedron, 4.6.8**, and the **truncated icosidodecahedron, 4.5.10**, which therefore need just 3 colours.

Interesting cases are the **snub cube,** $\mathbf{3^4.4}$ and **snub dodecahedron,** $\mathbf{3^4.5}$. Take the first. Of the 4 triangles around each vertex one does not touch the square. Since no square touches another, all the squares and this particular set of triangles can receive the same colour. This network isolates triangle pairs (bent rhombuses). These need just 2 colours. As a result only 3 colours are needed for the solid as a whole. Substituting 'pentagon' for 'square', the same applies to the snub dodecahedron:

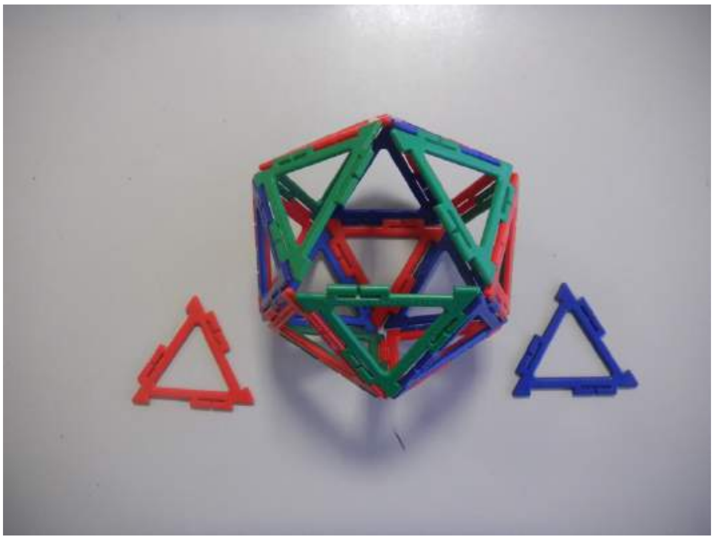


In the case of the **rhombicuboctahedron, 3.4^3** one of the squares around a vertex does not touch triangles. The triangles are isolated from each other so, again, both sets of faces can receive the one colour. Into the gaps in this network fall the remaining squares, which receive a second colour, so the solid only needs 2.

One solid remains to be dealt with, the **icosahedron, 3^5**. As with antiprisms, a triangle's neighbours do not touch. Here we draw the net, looking down on a rotation symmetry axis of order 3, and try to preserve that symmetry in our colours:



The white arrows show the vertices which become one when the net is closed. The black lines show the edges of the final face which we must insert:



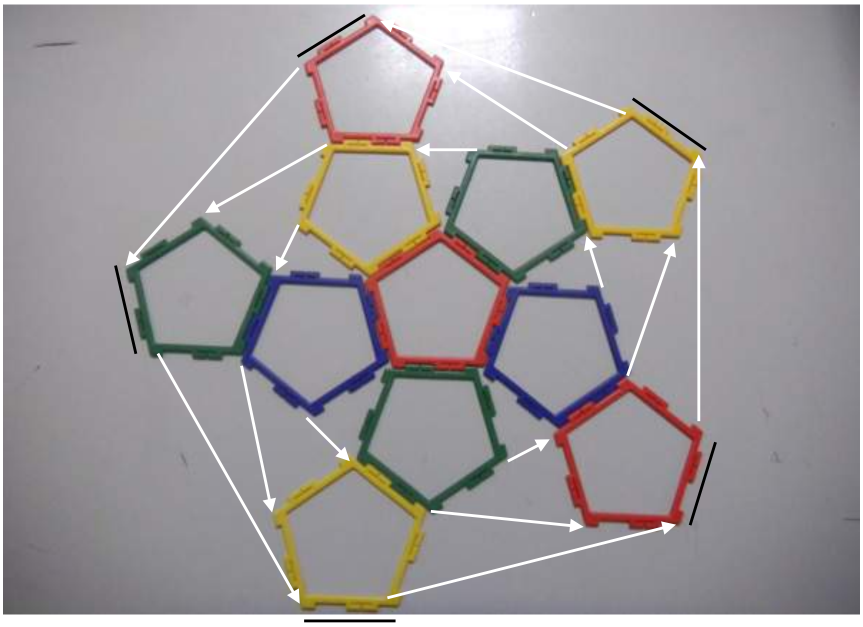
As you see, we have the choice of red, giving this number of each colour:

Red	Blue	Green
8	6	6

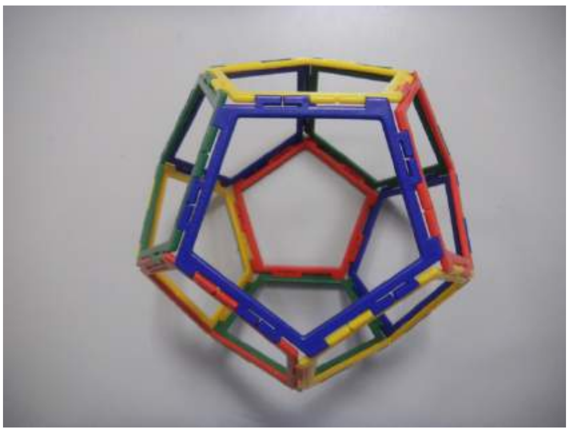
… or blue, giving this number of each colour:

Red	Blue	Green
7	7	6

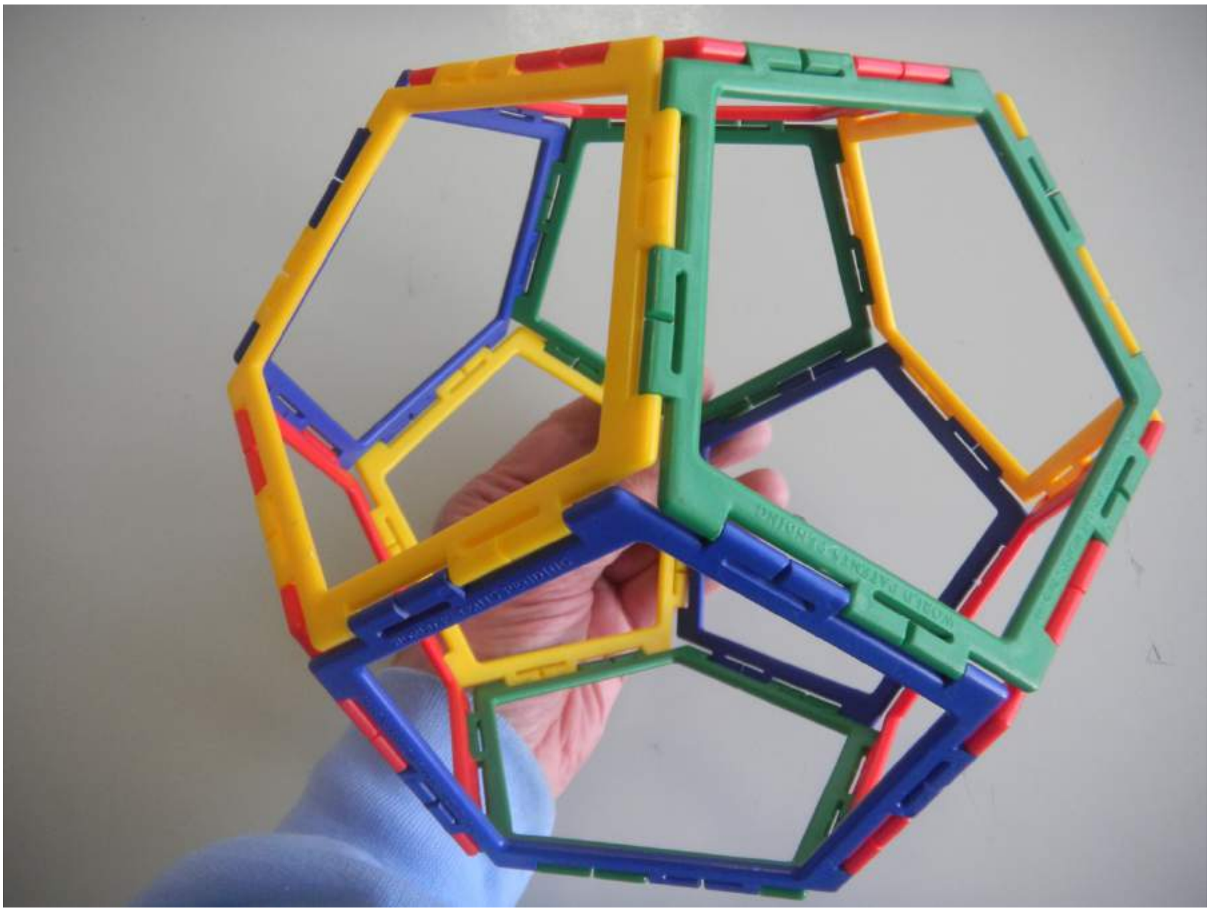
Having decided on the number of colours needed, it may still not be easy to arrange them correctly. An example is one of the simplest solids, the regular dodecahedron. As with the icosahedron, it's useful to draw the net looking down on a rotation symmetry axis passing through the centre of a face, in this case one of order 5:



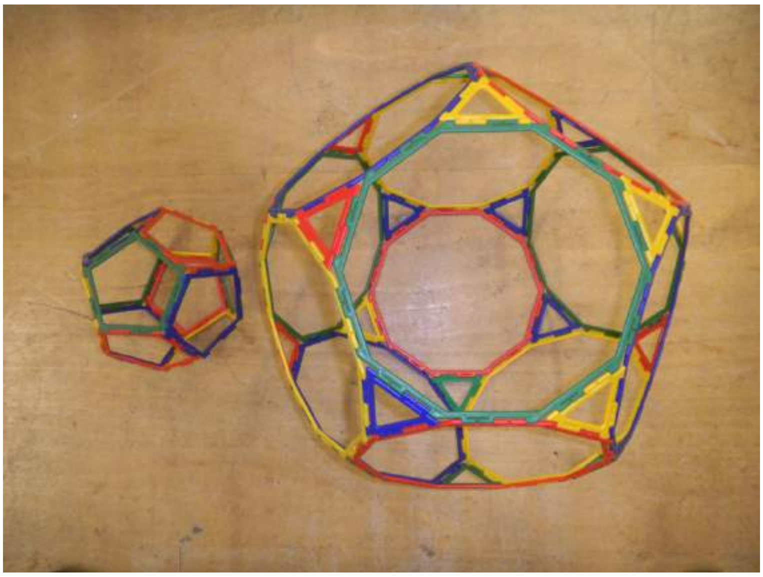
Moving outwards from the central pentagon, we alternate blue and green in the first ring, requiring yellow to close the circuit. What we must then do is alternate the other pair of colours, red and yellow, in the outer ring, ensuring that a yellow face does not meet the yellow pentagon in the inner ring. Again we need a third colour, green, to close the circuit. But that is OK because, as shown by the black lines, that only uses up 3 colours, leaving available a 4th, blue, to close the solid:



12 faces, 4 colours, 3 of each. As you would expect, they are symmetrically disposed about a 3-fold axis of rotation symmetry. The photograph shows the axis about which the red faces are symmetrically arranged:



One of the nicest features forced by symmetry is the disposal of triangles about the decagonal faces in the truncated dodecahedron. The picture shows the truncated form beside the original:



The colour scheme means there must be 3 consecutive triangles of colour *A* and 1 triangle each of colours *B* and *C* around a decagon of colour *D*.

Paul Stephenson, 5.2.17

Post script

In my piece ‘The Magic Mathworks Travelling Circus and Polydron’ I showed a net of the snub dodecahedron using 4 colours. Only recently did I realise that (as explained in this article) 3 are enough.